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A dual description of the four dimensional non-linear sigma model

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Abstract

The dual of the four dimensional non-linear sigma model is constructed using techniques familiar to string theory. This construction necessitates the introduction of a rank two anti-symmetric tensor field whose properties are examined. The physics of the dual theory and that of the original model are compared. As an illustration we study in detail the SU(2) chiral model. We find that the scattering amplitudes of the charged Goldstone bosons in the two theories are in complete agreement at the one loop level.

1. Introduction

One of the most striking features of string theory is that it possesses a dual symmetry which typically reveals itself in two ways. The first is known as "T duality" and is a generic feature of theories with compactified spatial dimensions. The simplest example of such duality manifests itself in the fact that the spectrum obtained when one dimension is compactified on a circle of radius R is found to be indistinguishable from that obtained when the compactification takes place on a circle of radius $1/R$ [1]. The second type of duality is known as "S duality" and interchanges the strong and weak coupling limits of string theory. This remarkably generalises the strong-weak (or electric-magnetic) duality conjuctured many years ago by Olive and Montonen [2] and shown by Osborn [3] to hold (if at all) for $N = 4$ supersymmetric gauge theories only.

Recently, dramatic new evidence for the validity of this conjecture has emerged from the work of A. Sen [4] and furthermore a version of Olive-Montonen duality was surprisingly found by Seiberg and Witten in $N = 2$ supersymmetric gauge theory in four dimensions [5].

These two dualities are, however, discrete symmetries of the spectrum of string theory. The question which we would like to ask is can this duality be implemented at the level of the two dimensional non-linear sigma model which describes the low energy theory of strings? Indeed, the dual theory of an arbitrary sigma model with an Abelian isometry was constructed by Buscher in ref.[6]. There is in fact an algorithm for constructing the dual theory of a sigma model which has the advantage of being applicable to sigma models in any number of dimensions [7]. The algorithm consists of gauging a symmetry (an isometry) of the action by introducing non-propagating gauge fields and whose field strength is forced to vanish by means of a Lagrange multiplier - the original theory being indeed regained if one integrates out the Lagrange multiplier. On the other hand, if one integrates over the gauge fields instead, one obtains the dual theory where the Lagrange multiplier is now a dynamical field.

It is this algorithm which we want to apply to a general four dimensional sigma model where in this case the Lagrange multiplier is a rank two antisymmetric field and needs a careful treatment.

Four dimensional sigma models, although non-renormalisable, are of great importance in the phenomenology of particle physics on account of the fact that they describe the dynamics of pions (and mesons in general) at energies which are small compared with the inverse confinement radius of QCD. The Higgs sector of the Standard Model is another example

where four dimensional sigma models play a crucial role, since in the limit of a very large Higgs mass, the Goldstone bosons of the symmetry breaking mechanism may be described by a non-linear sigma model as discussed in [8, 9, 10]. The connection in this case is made via the equivalence theorem [8, 11, 12, 13] which states that, for a given energy regime, the high energy amplitude for a process with external longitudinally polarised vector bosons is equal to the amplitude of the process in which the external vector bosons are replaced by the corresponding Goldstone bosons. Therefore the study of the dual of the four dimensional sigma model might have some physical significance in the Standard Model.

In this paper we would like to examine the dual theory of the four dimensional non-linear sigma model. This study requires a good understanding of the gauge theory of a rank two antisymmetric tensor field and we therefore start, in section two, by reviewing this theory and showing explicitly how the antisymmetric field could be taken as the dual of a scalar field. Section three deals with the construction of the dual theory of a four dimensional sigma model, which is in fact analogous to the one used in two dimensional sigma models. We apply this construction to the $SU(2)$ chiral sigma model in section four where we calculate the scattering amplitudes at one loop in the dual theory and compare them with those obtained from the original theory. Finally we end this paper with some comments and highlight further issues which need to be explored.

2. An alternative description of a scalar field

As will be made clear shortly, an idea central to our duality programme will be the replacement of a scalar field, ϕ, with a four dimensional antisymmetric tensor field, $\lambda_{\mu\nu}$. We immediately have a problem with the degrees of freedom count - the scalar field has just one, whilst the antisymmetric tensor field has six. It is therefore very instructive and pedagogical to analyse in some detail the gauge theory of an antisymmetric tensor field. Our starting point is the Lagrangian of this theory which we take to have the form [14]

$$\mathcal{L} = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\partial_\nu\lambda_{\rho\sigma}\epsilon_{\mu\nu'\rho'\sigma'}\partial_{\nu'}\lambda_{\rho'\sigma'}. \tag{1}$$

To count the degrees of freedom we can simply use phase space to identify the true degrees of freedom, however we prefer to use a covariant way of doing the counting.

Using the identity

$$\epsilon_{\mu\nu\rho\sigma}\epsilon_{\mu\nu'\rho'\sigma'} = g_{\nu\nu'}(g_{\rho\rho'}g_{\sigma\sigma'} - g_{\rho\sigma'}g_{\sigma\rho'}) + g_{\rho\nu'}(g_{\sigma\rho'}g_{\nu\sigma'} - g_{\sigma\sigma'}g_{\nu\rho'}) + g_{\sigma\nu'}(g_{\nu\rho'}g_{\rho\sigma'} - g_{\nu\sigma'}g_{\rho\rho'}) \tag{2}$$

we obtain the following equation of motion for the antisymmetric field $\lambda_{\rho\sigma}$

$$\Box\lambda_{\rho\sigma} - \partial_\nu \left(\partial_\rho \lambda_{\nu\sigma} - \partial_\sigma \lambda_{\nu\rho}\right) = 0. \tag{3}$$

These equations (and the action) are left unchanged if we perform the gauge transformation

$$\lambda_{\rho\sigma} \to \lambda'_{\rho\sigma} = \lambda_{\rho\sigma} + \partial_{[\rho}\zeta_{\sigma]} \tag{4}$$

and using this freedom we choose our $\lambda_{\rho\sigma}$ to satsify (the Lorentz gauge)

$$\partial_\rho \lambda_{\rho\sigma} = 0. \tag{5}$$

This has the advantage that it decouples the different components of $\lambda_{\rho\sigma}$ in a covariant way whilst leaving us with the simple wave equation

$$\Box\lambda_{\rho\sigma} = 0 \tag{6}$$

which has plane wave solutions of the form[1]

$$\lambda_{\rho\sigma} = \mathcal{V}_{\rho\sigma} \exp(-ik.x) \tag{7}$$

where $k^2 = 0$, and $\mathcal{V}_{\rho\sigma}$ contains the polarisation information of the field. The gauge condition then leads to the constraints

$$k_\rho \mathcal{V}_{\rho\sigma} = 0. \tag{8}$$

This is a set of four equations of which only three are independent, hence the polarisation constraints eliminate three (out of six) degrees of freedom. To make the identification with a scalar field we must eliminate still two more degrees of freedom and the required condition comes from the further gauge freedom allowed by the constraint equation. In fact we have not yet exhausted the constraints imposed by gauge invariance. Within the Lorentz gauge, we are still free to make another gauge transformation

$$\lambda_{\rho\sigma} \to \lambda_{\rho\sigma} + \partial_{[\rho}A_{\sigma]} \tag{9}$$

provided we demand that A_ρ satisfies

$$\Box A_\sigma - \partial_\rho \partial_\sigma A_\rho = 0. \tag{10}$$

These last equations are simply Maxwell's equations for electromagnetism in the vacuum and are themselves left invariant under

$$A_\rho \to A_\rho + \partial_\rho \psi. \tag{11}$$

[1]Although (for simplicity) we are using "unquantised" language, the extension of the argument to the quantised field $\lambda_{\rho\sigma}$ is transparent.

We are therefore led to gauge fix the gauge fixing conditions themselves and this will have important consequences when the theory is quantised - it leads to what are known as "ghosts for ghosts".

As in electromagnetism we can choose the Lorentz gauge for A_μ such that

$$\partial_\mu A_\mu = 0. \tag{12}$$

leading once again to the wave equation, $\Box A_\mu = 0$, and the solution

$$A_\rho = \epsilon_\rho \exp(-ik.x) \tag{13}$$

with ϵ_ρ containing the polarisation information of A_μ and again $k^2 = 0$. The polarisation vector ϵ_μ is constrained by the gauge choice to satisfy

$$k_\mu \epsilon_\mu = 0. \tag{14}$$

This last equation eliminates one component of the gauge function A_μ. However, as is well-known in electromagnetism, the Lorentz gauge does not fix the gauge transformations completely since we remain in the Lorentz gauge by performing the gauge transformation

$$A_\rho \rightarrow A_\rho + \partial_\rho \chi \qquad \text{provided} \qquad \Box\chi = 0. \tag{15}$$

The condition $\Box\chi = 0$ also has a wave function solution $\chi = \exp(-ik.x)$. This residual gauge freedom corresponds to changing ϵ_μ by a multiple of k_μ

$$\epsilon_\mu \rightarrow \epsilon_\mu + \beta k_\mu \tag{16}$$

allowing us to kill another component of the gauge function A_μ, and so, as expected from electomagnetism, A_μ has only two independent polarisations.

Let us now go back to our gauge choice for $\lambda_{\mu\nu}$ and see the effects of all this on the polarisation tensor $\mathcal{V}_{\rho\sigma}$. For our plane wave solutions, the residual gauge transformation on the field $\lambda_{\mu\nu}$ amounts to the change

$$\mathcal{V}_{\rho\sigma} \rightarrow \mathcal{V}_{\rho\sigma} + (k_\rho \epsilon_\sigma - k_\sigma \epsilon_\rho) \quad . \tag{17}$$

Since ϵ_μ has only two independent components, this residual gauge transformation can be used to remove two more degrees of freedom. Therefore the overall change has been $6 \rightarrow 3 \rightarrow 1$ and our $\lambda_{\rho\sigma}$ now has the correct number of degrees of freedom to be seriously taken to represent a real scalar field.

The duality between the antisymmetric tensor field and the free scalar field can be made more plausible using a path integral formulation. Notice that the above action depends only on the "field strength" of $\lambda_{\mu\nu}$ namely $V_\mu = \epsilon_{\mu\nu\rho\sigma}\partial_\nu\lambda_{\rho\sigma}$. We have simply

$$\mathcal{L} = -\frac{1}{2}V_\mu V_\mu \tag{18}$$

subject to the constraint $\partial_\mu V_\mu = 0$. Including this restriction, the generating functional for this model is

$$\begin{aligned} \mathcal{Z} &= \int \mathcal{D}\lambda \exp i \left(-\frac{1}{2}g^2 \int \mathrm{d}^4x \epsilon_{\mu\nu\rho\sigma}\partial_\nu\lambda_{\rho\sigma}\epsilon_{\mu\nu'\rho'\sigma'}\partial_{\nu'}\lambda_{\rho'\sigma'}. \right) \\ &= \int \mathcal{D}V \delta\left(\partial_\mu V_\mu\right) \exp i \left(-\frac{1}{2}g^2 \int \mathrm{d}^4x V_\mu V_\mu \right) \quad . \end{aligned} \tag{19}$$

Here g is a dimensionful coupling constant and since the theory is free, g could be simply absorbed into the antisymmetric field $\lambda_{\mu\nu}$, however for later convenience we choose to keep it explicit. We implement the delta function as usual in the path integral with the consequent introduction of a Lagrange multiplier, ϕ, which can here be interpreted as a scalar field. We obtain

$$\begin{aligned} \mathcal{Z} &= \int \mathcal{D}V\mathcal{D}\phi \exp i \int \mathrm{d}^4x \left(-\frac{1}{2}g^2 V_\mu V_\mu + \phi\partial_\mu V_\mu \right) \\ &= \int \mathcal{D}V \exp -\frac{ig^2}{2} \int \mathrm{d}^4x \left(V_\mu + \frac{1}{g^2}\partial_\mu\phi \right) \left(V_\mu + \frac{1}{g^2}\partial_\mu\phi \right) \\ &\times \int \mathcal{D}\phi \exp \frac{i}{2g^2} \int \mathrm{d}^4x \left(\partial_\mu\phi\partial^\mu\phi\right). \end{aligned} \tag{20}$$

The path integral over V is now in the form of a Gaussian and is just a number which we normalise to 1 leaving

$$\mathcal{Z} = \int \mathcal{D}\phi \exp \frac{i}{2g^2}\partial^\mu\phi\partial_\mu\phi \tag{21}$$

which is of the correct form to describe a scalar field. (We note also that the scale factor, g, has become inverted).

In performing this manipulation we have glossed over an important point which we now bring to light[2]. The generating functional (19) in fact contains source terms and (in terms of $\lambda_{\mu\nu}$) we should have considered

$$\mathcal{Z}[J] = \int \mathcal{D}\lambda \exp i \int \mathrm{d}^4x \left[-\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\epsilon_{\mu\nu'\sigma'\rho'}\partial_\nu\lambda_{\rho\sigma}\partial_{\nu'}\lambda_{\rho'\sigma'} + J^{\rho\sigma}\lambda_{\rho\sigma} \right] \tag{22}$$

with $J^{\rho\sigma}$ an antisymmetric source. To be able to go from the path integral integration over $\lambda_{\mu\nu}$ to a path integral over V_μ demands that the source term should be restricted to have

[2]We thank D. A. Ross for pointing this out to us.

the form

$$\int d^4x J^{\rho\sigma}\lambda_{\rho\sigma} = \int d^4x \epsilon^{\mu\nu\rho\sigma}\partial_\mu\chi_\nu\lambda_{\rho\sigma} = -\int d^4x \chi_\nu\epsilon^{\mu\nu\rho\sigma}\partial_\mu\lambda_{\rho\sigma} = \int d^4x \chi^\mu V_\mu \tag{23}$$

to be of a suitable form to act as a source term for the V_μ fields.

We therefore see that before our fields can be interpreted as an alternative description of scalar fields the source term must have the very specific form

$$J^{\rho\sigma} = \epsilon^{\mu\nu\rho\sigma}\partial_\mu\chi_\nu \tag{24}$$

and this causes complications when the antisymmetric tensor field $\lambda_{\mu\nu}$ is no longer free.

Before leaving this section we would like to examine the quantisation of the antisymmetric field $\lambda_{\mu\nu}$ [15, 16]. We choose to quantise the theory in the Lorentz gauge $\partial_\mu\lambda_{\mu\nu} = 0$ which demands the addition of the gauge fixing term

$$\mathcal{L}_{\text{g.f}} = -2\partial_\mu\lambda_{\mu\alpha}\partial_\nu\lambda_{\nu\alpha} \tag{25}$$

to our starting action. The ghost Lagrangian corresponding to this gauge fixing is given by

$$\mathcal{L}_{\text{gh}} = C^*_\mu\left(\Box g_{\mu\nu} - \partial_\mu\partial_\nu\right)C_\nu. \tag{26}$$

which is itself invariant under

$$\delta C_\mu = \partial_\mu\alpha \ \ , \ \ \delta C^*_\mu = \partial_\mu\beta, \tag{27}$$

with α and β two anticommuting scalar functions. Since we are integrating over C_μ and C^*_μ in the path integral, we need to fix the ghost gauge freedom, i.e. we shall be adding ghosts for ghosts. We fix the gauge by introducing the term

$$\mathcal{L}_{\text{gh.g.f}} = -\partial^\mu C^*_\mu\partial^\nu C_\nu \tag{28}$$

and add a corresponding ghost Lagrangian for this new gauge fixing

$$\mathcal{L}_{\text{gh.gh}} = \xi^*\Box\xi + \kappa^*\Box\kappa \ \ , \tag{29}$$

where ξ and κ are commuting scalars and are the ghosts for ghost fields. Hence the total effective Lagrangian is given by

$$\mathcal{L}_{\text{eff}} = \lambda_{\mu\nu}\Box\lambda_{\mu\nu} + C^*_\mu\Box C_\mu + \xi^*\Box\xi + \kappa^*\Box\kappa \ \ . \tag{30}$$

The quantisation of the free antisymmetric field will be of use in the following sections.

3. The Abelian Duality

In this section we would like to apply the techniques of string theory in constructing the dual theory of the four dimensional sigma model. Let us take a target space whose coordinates, ρ^a, we split as $\rho^a = (\theta, \phi^i)$ and, without loss of generality, define a four dimensional non-linear sigma model on this space as

$$S(\theta,\phi) = \int d^4x \left[\frac{1}{2}G(\phi)\,\partial_\mu\theta\partial_\mu\theta + G_i(\phi)\,\partial_\mu\phi^i\partial_\mu\theta + \frac{1}{2}g_{ij}(\phi)\,\partial_\mu\phi^i\partial_\mu\phi^j\right] \quad . \tag{31}$$

This action is invariant under the global transformation $\theta \to \theta + \alpha$ and the duality transformation emerges upon minimally gauging this global symmetry and adding a Lagrange multiplier term constraining the gauge field to be pure gauge [7]. We therefore consider

$$S(\theta,\phi) = \int d^4x \left[\frac{1}{2}G(\phi)\,D_\mu\theta D_\mu\theta + G_i(\phi)\,\partial_\mu\phi^i D_\mu\theta + \frac{1}{2}g_{ij}(\phi)\,\partial_\mu\phi^i\partial_\mu\phi^j - \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\lambda_{\rho\sigma}F_{\mu\nu}\right] \tag{32}$$

where $D_\mu\theta = \partial_\mu\theta + A_\mu$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Variation with respect to $\lambda_{\rho\sigma}$ imposes the constaint $F_{\mu\nu} = 0$ which is solved by $A_\mu = \partial_\mu\xi$ which in turn gives

$$D_\mu\theta = \partial_\mu\theta + A_\mu \;\; = \;\; \partial_\mu(\theta + \xi) \tag{33}$$

i.e. the effect of rewriting the model in a gauged form has been to replace θ by $\theta + \xi$ which is dynamically of no consequence. Hence eliminating the Lagrange multiplier takes us back to the original theory.

Next, keeping the Lagrange multiplier and varying instead with respect to the gauge field gives, after integration by parts,

$$\frac{\delta\mathcal{L}}{\delta A_\mu} = G(\partial_\mu\theta + A_\mu) + G_i\partial_\mu\phi^i - \epsilon_{\mu\nu\rho\sigma}\partial_\nu\lambda_{\rho\sigma} = 0 \tag{34}$$

or

$$D_\mu\theta = \frac{1}{G}\left(\epsilon_{\mu\nu\rho\sigma}\partial_\nu\lambda_{\rho\sigma} - G_i\partial_\mu\phi^i\right). \tag{35}$$

Substituting this back into (32) we obtain our form for the dual action

$$S(\lambda,\phi) = \int d^4x \left[\frac{1}{2}G_{ij}\partial_\mu\phi^i\partial^\mu\phi^j - \frac{1}{2G}\epsilon_{\mu\nu\rho\sigma}\partial_\nu\lambda_{\rho\sigma}\epsilon_{\mu\nu'\rho'\sigma'}\partial_{\nu'}\lambda_{\rho'\sigma'} + \frac{1}{G}\epsilon_{\mu\nu\rho\sigma}\partial_\nu\lambda_{\rho\sigma}G_i\partial_\mu\phi^i\right] \tag{36}$$

where G_{ij} is given by

$$G_{ij} = g_{ij} - \frac{1}{G}G_iG_j. \tag{37}$$

Therefore the dual action describes a non-linear sigma model with metric G_{ij} interacting with a dynamical antisymmetric tensor field. It could be thought that the antisymmetric

field $\lambda_{\mu\nu}$ has replaced the field θ in the original model. We also notice that the coupling G has become inverted in various terms and take this to imply that certain strong coupling behaviour will now be described correctly with techniques appropriate to small coupling as expected from duality.

The duality in the action $S(\lambda, \phi)$ can be seen from the fact that this action remains invariant under the interchange

$$\epsilon_{\mu\nu\rho\sigma}\partial_\nu\lambda_{\rho\sigma} \quad \leftrightarrow \quad G_i\partial_\mu\phi^i, \tag{38}$$

which is in the spirit of the duality encountered in electomagnetism between the electric and magnetic fields.

Another appealing feature of the dual action $S(\lambda, \phi)$ can be seen by casting it in the form

$$S(\lambda, \phi) = \int \mathrm{d}^4x \left[\frac{1}{2}G_{ij}\partial_\mu\phi^i\partial^\mu\phi^j - \frac{1}{2G}\epsilon_{\mu\nu\rho\sigma}\partial_\nu\lambda_{\rho\sigma}\epsilon_{\mu\nu'\rho'\sigma'}\partial_{\nu'}\lambda_{\rho'\sigma'} + \lambda_{\mu\nu}\Omega_{\mu\nu}\right] \tag{39}$$

where $\Omega_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma}\partial_\rho\left(\frac{1}{G}G_i\partial_\sigma\phi\right)$. In this form the model describes a hydrodynamical flow in the presence of a vortex $\Omega_{\mu\nu}$; the antisymmetric field $\lambda_{\mu\nu}$ is then the velocity potential and V_μ is the velocity vector satisfying the continuity equation $\partial_\mu V_\mu = 0$. A similar phenomenon has been shown to exist in the Abelian Higgs model by Sugamoto [17].

4. Application to SU(2)

We turn now to investigate some phenomenological implications of what we have done. For definiteness we restrict our analysis to the $SU(2)$ case although the generalisation to other Lie algebras is staightforward. We take as our starting point the non-linear sigma model parameterised by the matrix field $U(x)$ belonging to the quotient space $SU(2)_L \otimes SU(2)_R/SU(2)_{L+R}$,

$$U(x) = \exp\left(i\tau^a\xi_a/\Lambda\right), \tag{40}$$

where ξ^a, $a = 1, 2, 3$, are the Goldstone bosons associated with the symmetry breaking $SU(2)_L \otimes SU(2)_R \to SU(2)_{L+R}$, τ^a are the 2×2 Pauli matrices and Λ some energy scale. The current interest in such models stems from the fact that they can be used to investigate the symmetry breaking sector of the Standard Model - the connection being made via the equivalence theorem which relates the scattering of longitudinally polarised weak vector bosons to those involving the Goldstone bosons associated with the above symmetry breaking pattern - these points are lucidly discussed in [18].

Under an $SU(2)_L \otimes SU(2)_R$ transformation the matrix U transforms as $U \to LUR^\dagger$ and the $SU(2)$ invariant sigma model can be written

$$\mathcal{L} = \frac{\Lambda^2}{4} \mathrm{Tr} \partial_\mu U \partial^\mu U^{-1} \tag{41}$$

where we let Λ take any value. (When applied to the symmetry breaking sector of the Standard Model Λ becomes fixed at the Higgs VEV scale of 246 GeV). Our interest is in the dual version of the model and in order to get this we must first massage (41) into the form (31) which we achieve by separating out one field - we choose that associated with τ^3 and write

$$U(x) = \exp\left(i\theta\tau_3/\Lambda\right) \exp\left(i\tau^j\phi_j/\Lambda\right) \tag{42}$$

where j runs over the values 1 and 2 only. This new parametrisation can be thought of as a change of variables from ξ^a to θ and ϕ^i.

Using the closed form expression for the exponential of any linear combination of Pauli matrices

$$\exp\left(i\tau^\alpha\lambda_\alpha/\Lambda\right) = 1\!\!1 \cos\Omega + i\tau^\alpha\lambda_\alpha \frac{1}{\Lambda}\frac{\sin\Omega}{\Omega} \qquad \text{with} \qquad \Omega^2 = \frac{\lambda^\alpha\lambda_\alpha}{\Lambda^2} \tag{43}$$

we arrive at the following form of the $SU(2)$ non-linear sigma model

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}\partial^\mu\theta\partial_\mu\theta + \frac{1}{2}\frac{1}{\Lambda}\frac{\sin^2\Omega}{\Omega^2}\epsilon_{ij}\pi^j\partial^\mu\theta\partial_\mu\pi^i + \frac{1}{4}\frac{\sin^2\Omega}{\Omega^2}\delta_{ij}\partial^\mu\pi^i\partial_\mu\pi^j \\ &+ \frac{1}{8\Lambda^2}\frac{1}{\Omega^2}\left[1 - \frac{\sin^2\Omega}{\Omega^2}\right]\delta_{ij}\delta_{kl}\pi^i\pi^k\partial^\mu\pi^j\partial_\mu\pi^l \end{aligned} \tag{44}$$

where we have defined $\pi^\pm = (\phi_1 \pm i\phi_2)$ and $\Omega^2 = \pi^+\pi^-/\Lambda^2$. The target space indices are contacted by the delta function, $\delta_{+-} = 1$, and $\epsilon_{+-} = -\epsilon_{-+} = i$. This action can be seen to be in the form of (31) with the identifications

$$\begin{aligned} g_{ij} &= \frac{1}{2}\frac{\sin^2\Omega}{\Omega^2}\delta_{ij} + \frac{1}{4}\frac{1}{\Omega^2\Lambda^2}\left[1 - \frac{\sin^2\Omega}{\Omega^2}\right]\delta_{ik}\delta_{jl}\pi^k\pi^l \\ G_i &= \frac{1}{2}\frac{1}{\Lambda}\epsilon_{ij}\pi^j\frac{\sin^2\Omega}{\Omega^2} \ , \ G = 1 \ . \end{aligned} \tag{45}$$

It is well known that in two dimensions this model is renormalisable in the sense that the counterterms can be absorbed into the terms already present in the tree level Lagrangian. In four dimensions, however, this is no longer true. To get around this problem we take (44) to be the first term in a general momentum expansion with an infinte number of terms and an infinite number of arbitrary parameters. The addition of these terms is necessary in order to absorb the higher dimensional divergences which one encounters in four dimensional sigma models. Now when we calculate divergent quantities we can renormalise the higher order

coefficients thereby making the theory finite - i.e. the higher order terms are demanded if the theory is to make sense.

Our aim is to calculate two-particle scattering amplitudes of the sigma model in the parametrisation (42) and to make a comparison with the results obtained using the dual theory. For this we enlarge (44) to include $O(p^4)$ terms and expand the principal sigma model up to four point interactions. We have then the Lagrangian obtained by expanding the principal chiral sigma model up to four fields

$$\mathcal{L} = +\frac{1}{2}\partial_\mu\theta\partial^\mu\theta+\frac{1}{4}\delta_{ij}\partial_\mu\pi^i\partial^\mu\pi^j+\frac{1}{2}\frac{1}{\Lambda}\epsilon_{ij}\pi^j\partial_\mu\theta\partial^\mu\pi^i-\frac{1}{24\Lambda^2}\left[\delta_{ij}\delta_{lm}-\delta_{il}\delta_{jm}\right]\pi^i\pi^j\partial_\mu\pi^l\partial^\mu\pi^m \quad (46)$$

plus the counterterm Lagrangian

$$\begin{aligned}\mathcal{L}_{\text{c.t.}} &= \frac{4}{\Lambda^4}(M+N)\partial_\mu\theta\partial^\mu\theta\partial_\nu\theta\partial^\nu\theta + \frac{8}{\Lambda^4}\delta_{ij}\left[M\partial_\mu\theta\partial^\mu\theta\partial_\nu\pi^i\partial^\nu\pi^j + N\partial_\mu\theta\partial^\mu\pi^i\partial_\nu\theta\partial^\nu\pi^j\right] \\ &+\frac{4}{\Lambda^4}\left[M\delta_{ij}\delta_{lm}+N\delta_{il}\delta_{jm}\right]\partial_\mu\pi^i\partial^\mu\pi^j\partial_\nu\pi^l\partial^\nu\pi^m \qquad (47)\end{aligned}$$

where M and N are the arbitrary coefficients of the $O(p^4)$ contributions. We stop at just four point terms since the processes of greatest phenomenological interest are the two-particle scattering amplitudes on account of the fact that the equivalence theorem directly relates these amplitudes to the amplitudes for scattering processes of the form $W^i_L W^j_L \to W^m_L W^n_L$ where $W^i = W^\pm,\ Z^0$. To one loop we need to consider 24 Feynman diagrams (compared with just three in the more conventional parametrisation (40)) a representative selection of which are given in fig.1. We obtain

$$\begin{aligned}\mathcal{M}\left(\pi^+\pi^-\to\pi^+\pi^-\right) &= -i\frac{u}{\Lambda^2}+\frac{4i}{\Lambda^4}\left[2M_R(s^2+t^2)+N_R(s^2+t^2+2u^2)\right] \\ &-\frac{i}{(4\pi)^2\Lambda^4}\left(\frac{1}{12}(9s^2+u^2-t^2)\ln\frac{-s}{\mu^2}\right. \\ &\left.+\frac{1}{12}(9t^2+u^2-s^2)\ln\frac{-t}{\mu^2}+\frac{1}{2}u^2\ln\frac{-u}{\mu^2}\right)\end{aligned}$$

$$\begin{aligned}\mathcal{M}\left(\pi^+\pi^-\to\theta\theta\right) &= i\frac{s}{\Lambda^2}+\frac{4i}{\Lambda^4}\left[2M_Rs^2+N_R(t^2+u^2)\right] \\ &-\frac{i}{(4\pi)^2\Lambda^4}\left(\frac{1}{12}(3t^2+u^2-s^2)\ln\frac{-t}{\mu^2}\right. \\ &\left.+\frac{1}{12}(3u^2+t^2-s^2)\ln\frac{-u}{\mu^2}+\frac{1}{2}s^2\ln\frac{-s}{\mu^2}\right)\end{aligned}$$

$$\mathcal{M}\left(\theta\theta \to \theta\theta\right) = \frac{8i}{\Lambda^4}\left[M_R + N_R\right]\left(s^2 + t^2 + u^2\right) + \frac{i}{(4\pi)^2\Lambda^4)}\left(-s^2 \ln\frac{-s}{\mu^2} - t^2 \ln\frac{-t}{\mu^2} - u^2 \ln\frac{-u}{\mu^2}\right). \quad (48)$$

Here s, t and u are the usual Mandelstam variables, μ is an arbitrary renormalisation scale and using dimensional regularisation in $d = 4 - 2\epsilon$ dimensions we have the renormalised forms of M and N (using modified minimal subtraction, $\overline{MS}$):

$$\begin{aligned} M_R &= M + \frac{1}{24}\frac{1}{(4\pi)^2}\left(\frac{1}{\epsilon} - \gamma + \ln(4\pi)\right) \\ N_R &= N + \frac{1}{12}\frac{1}{(4\pi)^2}\left(\frac{1}{\epsilon} - \gamma + \ln(4\pi)\right). \end{aligned} \quad (49)$$

These scattering amplitudes are in precise agreement with the results presented in [18].

Let us now turn to the calculation of the equivalent scattering amplitudes in the dual model. Using the form of the metric elements given in (45) and using (36) we can move directly to the dual description and consider the Lagrangian

$$\begin{aligned} \mathcal{L}^D &= -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\partial_\nu\lambda_{\rho\sigma}\epsilon_{\mu\nu'\rho'\sigma'}\partial_{\nu'}\lambda_{\rho'\sigma'} + \frac{1}{2}\frac{1}{\Lambda}\epsilon_{\mu\nu\rho\sigma}\partial_\nu\lambda_{\rho\sigma}\epsilon_{ij}\pi^j\partial^\mu\pi^i\frac{\sin^2\Omega}{\Omega^2} \\ &+ \frac{1}{4}\partial^\mu\pi.\partial_\mu\pi\frac{\sin^2\Omega}{\Omega^2}\left(1 - \sin^2\Omega\right) + \frac{1}{8}\frac{1}{\Lambda^2\Omega^2}\left[1 - \frac{\sin^2\Omega}{\Omega^2} + \frac{\sin^4\Omega}{\Omega^2}\right]\left(\pi.\partial_\mu\pi\right)^2. \end{aligned} \quad (50)$$

Expanding to four point terms we have

$$\begin{aligned} \mathcal{L} &= -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\partial_\nu\lambda_{\rho\sigma}\epsilon_{\mu\nu'\rho'\sigma'}\partial_{\nu'}\lambda_{\rho'\sigma'} + \frac{1}{2}\frac{1}{\Lambda}\epsilon_{\mu\nu\rho\sigma}\epsilon_{ij}\pi^j\partial_\nu\lambda_{\rho\sigma}\partial^\mu\pi^i \\ &+ \frac{1}{4}\partial^\mu\pi^i\partial_\mu\pi^i + \frac{1}{6}\frac{1}{\Lambda^2}\left(\delta_{ik}\delta_{jl} - \delta_{ij}\delta_{kl}\right)\pi^i\pi^j\partial^\mu\pi^k\partial_\mu\pi^l \end{aligned} \quad (51)$$

giving at tree level the independent matrix elements

$$\mathcal{M}\left(\pi^+\pi^- \to \pi^+\pi^-\right) = -i\frac{u}{\Lambda^2}, \qquad \mathcal{M}\left(\pi^+\pi^- \to \lambda_{\mu\nu}\lambda_{\rho\sigma}\right) = i\frac{s}{\Lambda^2}$$
$$\mathcal{M}\left(\lambda_{\mu\nu}\lambda_{\rho\sigma} \to \lambda_{\mu'\nu'}\lambda_{\rho'\sigma'}\right) = 0 \quad (52)$$

in precise agreement with the original model. To get this we have used the sum over polarisation vectors, $\mathcal{V}_{\rho\sigma}$, for external $\lambda_{\rho\sigma}$ fields

$$\sum_\varepsilon \mathcal{V}_{\rho\sigma}(\varepsilon)\mathcal{V}^*_{\rho'\sigma'}(\varepsilon) = \frac{1}{4}\left(g_{\rho\rho'}g_{\sigma\sigma'} - g_{\rho\sigma'}g_{\sigma\rho'}\right) \quad (53)$$

where ε labels the helicity of the lines. At tree level, therefore, we are allowed to make the identification that the one physical degree of freedom associated with $\lambda_{\rho\sigma}$ is the Goldstone boson Z^0.

Proceeding to the higher order corrections we have calculated the scattering amplitudes for the charged pions in the dual model. The Feynman diagrams which need to be considered in this case are given in fig.2 *and* fig.3, where we note in particular the non-zero contribution of the one-particle reducible diagrams. The net result due to these diagrams is given by

$$\begin{aligned}
\mathcal{M}\left(\pi^+\pi^- \to \pi^+\pi^-\right) &= -i\frac{u}{\Lambda^2} + \frac{4i}{\Lambda^4}\left[2M_R(s^2+t^2) + N_R(s^2+t^2+2u^2)\right] \\
&\quad - \frac{i}{(4\pi)^2\Lambda^4}\left(\frac{1}{12}(9s^2+u^2-t^2)\ln\frac{-s}{\mu^2}\right. \\
&\quad \left. + \frac{1}{12}(9t^2+u^2-s^2)\ln\frac{-t}{\mu^2} + \frac{1}{2}u^2\ln\frac{-u}{\mu^2}\right)
\end{aligned}$$

$$\begin{aligned}
\mathcal{M}\left(\pi^+\pi^- \to \lambda\lambda\right) &= i\frac{s}{\Lambda^2} + \frac{4i}{\Lambda^4}\left[2M_R s^2 + N_R(t^2+u^2)\right] \\
&\quad - \frac{i}{(4\pi)^2\Lambda^4}\left(\frac{1}{12}(3t^2+u^2-s^2)\ln\frac{-t}{\mu^2}\right. \\
&\quad \left. + \frac{1}{12}(3u^2+t^2-s^2)\ln\frac{-u}{\mu^2} + \frac{1}{2}s^2\ln\frac{-s}{\mu^2}\right)
\end{aligned} \tag{54}$$

and again these expressions are precisely those found in the original model. Hence, to one loop, the scattering amplitudes of the charged pions in both model do agree. The scattering amplitude for four external antisymmetric tensor fields is very much more involved and will be treated elsewhere.

Notice that these scattering amplitudes involve the renormalised parameters M_R and N_R as given in (49). This means that we have actually added counterterms of dimension four to the dual theory. These terms are found by taking (47) and replacing $\partial_\mu\theta$ by $\epsilon_{\mu\nu\rho\sigma}\partial_\nu\lambda_{\rho\sigma}$.

5. Conclusions

We have studied, at the quantum level, the Abelian gauge theory of a rank two antisymmetric tensor field non-trivially interacting with scalar fields in the form of a non-linear sigma model. This theory is the dual theory of a four dimensional sigma model obtained using techniques of two dimensional theories and we have shown that the scattering amplitudes of the charged pions are the same, at the one loop level, in the dual and the original theories.

It is therefore clear that the duality transformations of the four dimensional sigma model do not change the physics of the original theory. This is in contrast to the two dimensional

case where the geometry and the physics of the dual theory are completely changed. This change is in fact due to the presence of the Wess-Zumino-Witten term in two dimensions. The dual two dimensional sigma model would be trivial if one set the Wess-Zumino-Witten term to zero. The question we would like to address now is could the inclusion of the Wess-Zumino-Witten term in four dimensions be of any consequence to the physics of the pions?

If chiral Lagrangians are to be taken as effective theories of QCD then they ought to incorporate all relevant symmetries of QCD and the presence of a Wess-Zumino-Witten term is then essential for the preservation of the symmetries of QCD [19]. In four dimensions, and in the notation of equation (31), this term can be written in the form

$$S_{\mathrm{wzw}}\left(\theta,\phi\right)=\int \mathrm{d}^4x\epsilon_{\mu\nu\rho\sigma}\left[b_{ijk}\left(\phi\right)\partial_\mu\theta\partial_\nu\phi^i\partial_\rho\phi^j\partial_\rho\phi^k+B_{ijkl}\left(\phi\right)\partial_\mu\phi^i\partial_\nu\phi^j\partial_\rho\phi^k\partial_\sigma\phi^l\right] \tag{55}$$

with b_{ijk} and B_{ijkl} totally antisymmetric tensors. (This demand for antisymmetry on the tensors makes it clear that for the case of $SU(2)$ no non-zero Wess-Zumino-Witten term can be generated). Adding this term to the sigma model action in (31) and performing the duality transformation leads to the action

$$I\left(\lambda,\phi\right)=\widetilde{S}\left(\theta,\phi\right)+\int \mathrm{d}^4x\epsilon_{\mu\nu\rho\sigma}B_{ijkl}\partial_\mu\phi^i\partial_\nu\phi^j\partial_\rho\phi^k\partial_\sigma\phi^l\,, \tag{56}$$

where $\widetilde{S}\left(\theta,\phi\right)$ is obtained from $S\left(\lambda,\phi\right)$ in (32) upon making the substitution

$$G_i\partial_\mu\phi^i \quad\rightarrow\quad G_i\partial_\mu\phi^i+\epsilon_{\mu\nu\rho\sigma}b_{ijk}\partial_\nu\phi^i\partial_\rho\phi^j\partial_\rho\phi^k\,. \tag{57}$$

Notice that terms of dimension four and six are generated in the dual theory - we certainly expect these higher dimensional terms to contribute beyond the one loop level. This issue is currently under investigation.

Another problem which is at the heart of dual theories is the infrared behaviour of these theories. It was shown in [20] that the inclusion of the Wess-Zumino-Witten term in the four dimensional sigma model leads to a non-trivial infrared fixed point, in addition to the usual Gaussian fixed point. This is a phenomenon that was shown to occur at three loops by analytically continuing the theory to a dimension less than four. We expect that the terms in the dual theory which are of dimension six will have dramatic consequences on the infrared behaviour of the theory at three loops.

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Representative one-loop diagrams referred to in the text

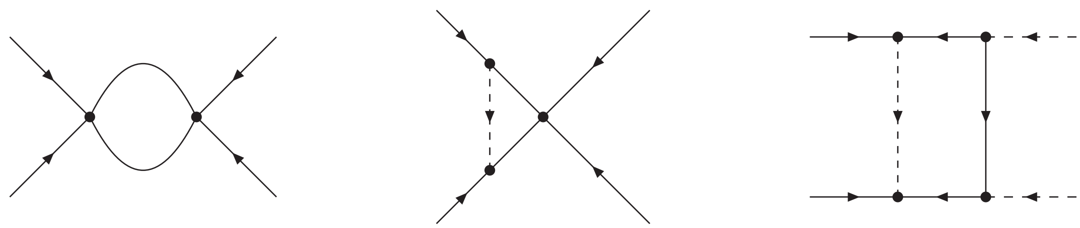

fig.1

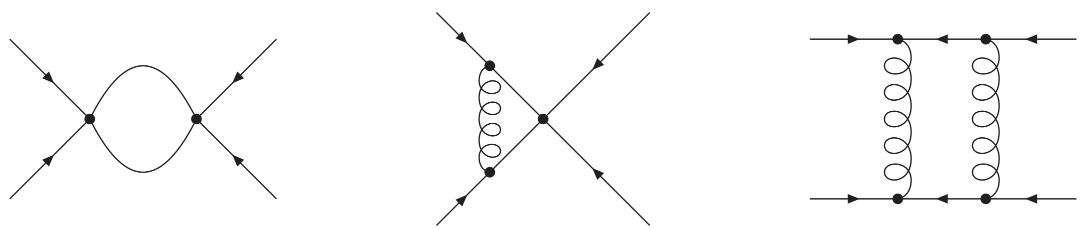

fig.2

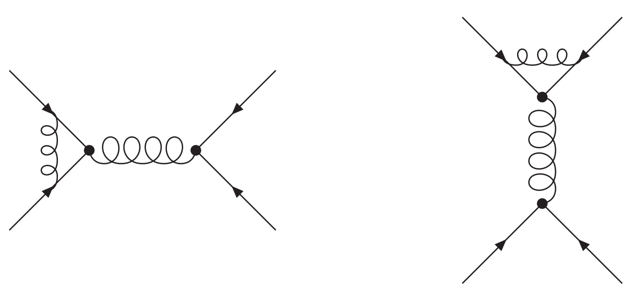

fig.3

Here the solid lines represent $\pi^{\pm}$ fields, the dashed lines represent the θ scalar field, whilst the coiled lines are the $\lambda_{\rho\sigma}$ antisymmetric tensor fields.

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